Some solutions for hwk #4

Section 2.1: #18b

Claim: Let A, B be sets. Then $A \subsetneq B \iff \mathcal{P}(A) \subsetneq \mathcal{P}(B)$.

Proof: [$\Rightarrow$] First suppose A, B are sets with $A \subsetneq B$.

Let $C \in \mathcal{P}(A)$, so $C \subseteq A$. Since $A \subseteq B$ we conclude $C \subseteq B$ so $C \in \mathcal{P}(B)$. Thus $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Now we must show $\mathcal{P}(A) \neq \mathcal{P}(B)$. Since $A \subsetneq B$, $\exists\ x \in B - A$. Then $\{x\} \subseteq B$ and $\{x\} \nsubseteq A$ so $\{x\} \in \mathcal{P}(B)$ but $\{x\} \notin \mathcal{P}(A)$. Thus $\mathcal{P}(B) \neq \mathcal{P}(A)$.

We conclude $\mathcal{P}(A) \subsetneq \mathcal{P}(B)$.

[$\Leftarrow$] Now assume A, B are sets with $\mathcal{P}(A) \subsetneq \mathcal{P}(B)$.

Since $A \subseteq A$ we know $A \in \mathcal{P}(A)$ so $\mathcal{P}(A) \subseteq \mathcal{P}(B) \rightarrow A \in \mathcal{P}(B)$ so $A \subseteq B$.

To prove that $A \neq B$ note that if $A = B$ then $\mathcal{P}(A) = \mathcal{P}(B)$, which is false.

Thus $A \subseteq B$ and $A \neq B$, so $A \subsetneq B$. ∎

Section 2.2 : # 11 a

Provide a counterexample for

$$A \cup C \subseteq B \cup C \rightarrow A \subseteq B.$$

<u>Counterexample</u>:

Let $A = \{1\}$, $B = \emptyset$, and $C = \{1, 2\}$.

The the hypothesis holds, since

$$A \cup C = \{1, 2\}$$

$$B \cup C = \{1, 2\}$$

but the conclusion does not, since $A \not\subseteq B$. //

Section 2.2: #12

Let A and B be sets.

(a) Prove that $\mathcal{P}(A\cap B) = \mathcal{P}(A)\cap\mathcal{P}(B)$ (using 9(c)).

Proof: Let A, B, C be sets.
Then $C\in\mathcal{P}(A\cap B) \iff C\subseteq A\cap B$ by definition.
By exercise 9(c) this is equivalent to
$C\subseteq A$ and $C\subseteq B$, which is true
if and only if $C\in\mathcal{P}(A)$ and $C\in\mathcal{P}(B)$, by
definition. This is equivalent to $C\in\mathcal{P}(A)\cap\mathcal{P}(B)$.

Thus
$$C\in\mathcal{P}(A\cap B) \iff C\in\mathcal{P}(A)\cap\mathcal{P}(B),$$
so $\mathcal{P}(A\cap B) = \mathcal{P}(A)\cap\mathcal{P}(B)$. ∎

(b) Prove $\mathcal{P}(A)\cup\mathcal{P}(B)\subseteq\mathcal{P}(A\cup B)$.

Proof: Let C be a set with $C\in\mathcal{P}(A)\cup\mathcal{P}(B)$.
Then either $C\in\mathcal{P}(A)$ or $C\in\mathcal{P}(B)$. Without loss
of generality assume $C\in\mathcal{P}(A)$, so $C\subseteq A$.
Since $A\subseteq A\cup B$ this implies $C\subseteq A\cup B$,
so $C\in\mathcal{P}(A\cup B)$. Thus
$$\mathcal{P}(A)\cup\mathcal{P}(B)\subseteq\mathcal{P}(A\cup B).$$ ∎

(c) Show the result in part (b) cannot be improved to equality.

Proof. We will construct a counterexample.
Let $A = \{1\}$ and $B = \{2\}$, so $A \cup B = \{1, 2\}$.
Then

$$\{1,2\} \in \mathcal{P}(A \cup B)$$

but

$$\{1,2\} \notin \mathcal{P}(A), \text{ and } \{1,2\} \notin \mathcal{P}(B), \text{ so}$$

$$\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B).$$

Thus $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$
for these sets, so it is not always true. ☑

Under what conditions is it true that

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)?$$

The equality holds if and only if

$$A \subseteq B \quad \text{or} \quad B \subseteq A.$$

(This is true because the sets in $\mathcal{P}(A \cup B)$ which are not in $\mathcal{P}(A)$ or $\mathcal{P}(B)$ are those containing elements in $B - A$ and $A - B$, so if either of those sets are empty then $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.)

I'll include a formal proof, even though the problem doesn't ask for it.

Claim: Let A, B be sets. Then

$$\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B) \iff A \subseteq B \text{ or } B \subseteq A.$$

Proof: [$\Leftarrow$] Without loss of generality assume $A \subseteq B$. We already know $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ from part (b) above so we will show the opposite inclusion. Since $A \subseteq B$ notice $A \cup B = B$ so

$$\mathcal{P}(A \cup B) = \mathcal{P}(B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B).$$

[$\Rightarrow$] We proceed by contrapositive. Suppose that A, B are sets such that it is not true that $A \subseteq B$ or $B \subseteq A$. Thus, we know $A \not\subseteq B$ and $B \not\subseteq A$. Then $B - A$ and $A - B$ are both nonempty sets. Let $x \in B - A$, $y \in A - B$.

Then $x \in B$ so $x \in A \cup B$ and $y \in A$ so $y \in A \cup B$. Thus $\{x, y\} \subseteq A \cup B$ so $\{x, y\} \in \mathcal{P}(A \cup B)$.

but $x \notin A$ so $\{x, y\} \not\subseteq A$ so $\{x, y\} \notin \mathcal{P}(A)$ and $y \notin B$ so $\{x, y\} \not\subseteq B$ so $\{x, y\} \notin \mathcal{P}(B)$.

Thus $\{x, y\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

So $\{x,y\} \in \mathcal{P}(A \cup B)$ but

$\{x,y\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$

so $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$. ∎

(d) Determine if $\mathcal{P}(A) \cap \mathcal{P}(B) = \emptyset$.

<u>Claim</u>: Let A, B be any two sets. Then $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are <u>not</u> disjoint.

<u>Proof</u>: Let A, B be any sets. Then $\emptyset \subseteq A$ and $\emptyset \subseteq B$ so

$$\emptyset \in \mathcal{P}(A) \cap \mathcal{P}(B).$$

Thus $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are not disjoint. ∎